Lecture 1 - Introduction

Microwave Active Circuit Analysis and Design

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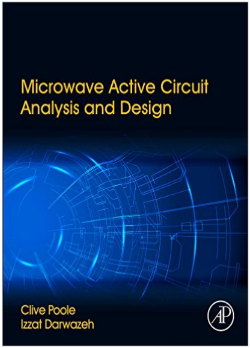

Intended Learning Outcomes

- Knowledge
  - Understand the characteristics that distinguish high frequency circuit design from low frequency circuit design
  - Be familiar with some basic EM theory and understand the importance of Maxwell's equations.
  - Understand some important properties of materials at RF frequencies (Permittivity and Permeability)
  - Understand that parasitic reactances associated with familiar lumped element components become more pronounced at radio frequencies and will significantly affect their impedance as a function of frequency, and that equivalent circuit models must therefore be used to adequately represent such components.
  - Revise the concept of quality factor, Q, for components and for the components when used in resonant circuits, and specifically its application to microwave resonators.
  - Become acquainted with the concept of *maximum power transfer*.
- Skills
  - Be able to calculate the inductance of a cylindrical wire.
  - Be able to design a single layer spiral inductor for a given inductance.
  - Be able to calculate the Q of common parallel and series RLC circuits.
  - Be able to determine the Q of a generic microwave resonator, based on return loss measurements.

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What are microwaves?

1. The term **microwave** commonly refers to the region of the electromagnetic spectrum from 1 GHz to 30 GHz (or 30 centimetres to 1 centimetre wavelengths).
2. The electrical properties of electronic components and interconnections depend upon their physical **Size** and **Geometry** in relation to the **Wavelength** of the signal being processed.
3. The physical Distance between components (i.e. the length of connecting wires) also becomes important when these distances are comparable to the wavelength of the signal. At microwave frequencies, connecting wires need to be considered as **Transmission Lines**, where Voltage and current at any instant are different at different positions along the line.
4. The behaviour of traditional passive circuit elements (R, L, C) departs from their ideals at microwave frequencies due to **Parasitic** effects.
5. At microwave frequencies reactive components (L and C) are often synthesised using transmission line sections.

The importance of radio frequency electronics

Many microwave circuit applications relate to portable, battery operated equipment, with the modern cellular phone handset being the most obvious example. The microwave design engineer will therefore be faced with a number of challenges and trade-offs, aside from the task of making the microwave circuit behave as intended. Figure 1 shows some of these trade-offs.

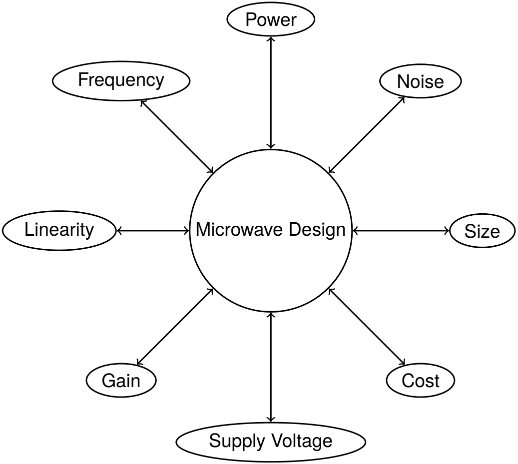

Figure 1 : RF design constraints and trade-offs

Electromagnetism basics

Let i, j, k be the corresponding basis of unit vectors. The divergence of a continuously differentiable vector field $E = Ui + Vj + Wk$ is equal to the scalar function:

$$\nabla \cdot \boldsymbol{E} = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} \qquad (1)$$

So $\nabla \cdot \boldsymbol{E}$, being a scalar, simply quantifies the amount of variation there is in the field, E. By contrast, $\nabla \times \boldsymbol{E}$ (pronounced 'curl E') measures how much E 'curls around', or how much it changes in the perpendicular directions.

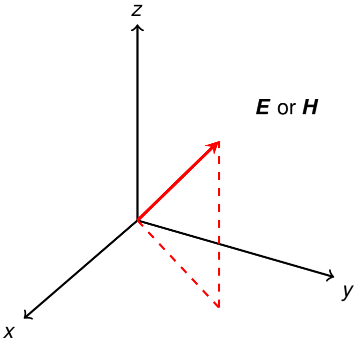

Figure 2 : Three dimensional coordinate space

This function is formally defined, for the vector field $E = Ui + Vj + Wk$, as:

$$\nabla \times \boldsymbol{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\mathbf{k} \qquad (2)$$

Maxwell's Equations

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0} \qquad (3)$$

$$\nabla \cdot \boldsymbol{B} = 0 \qquad (4)$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad (5)$$

$$\nabla \times \boldsymbol{B} = \mu_0 \left(\boldsymbol{J} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right) \qquad (6)$$

we can put the four equations of (3) to (6), in context as follows :

1. Equation 3 - (Gauss's law) : Electric charges create electric fields where the strength of the field is determined by the distance from the charge.
2. Equation 4 - (Gauss's law of magnetism) : There are no magnetic monopoles - the magentic field flux through any Gaussian surface sums to zero.
3. Equation 5 - (Faraday's Law) : A change in magnetic field strength induces a change in electric field strength.
4. Equation 6 - (Ampere's Law plus correction) : Faraday's Law reversed, plus electric current also creates magnetic fields.

Electromagnetism basics

By means of these four equations, Maxwell demonstrated that electric and magnetic forces are two complementary aspects of a single phenomenon now known as *electromagnetism*.

In a free space region with no charges ($\rho = 0$) and no currents ($\boldsymbol{J} = 0$), the equations (3) to (6), reduce to:

$$\nabla \cdot \boldsymbol{E} = 0 \quad (7)$$

$$\nabla \cdot \boldsymbol{B} = 0 \quad (8)$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \quad (9)$$

$$\nabla \times \boldsymbol{B} = \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \quad (10)$$

Electromagnetism basics

Taking the curl ($\nabla\times$) of the curl equations, and applying a theorem in vector calculus known as the 'curl of the curl' identity, (i.e. $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$), we obtain the following 'wave equations' in three dimensions:

$$\mu_0\varepsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} - \nabla^2 \boldsymbol{E} = 0 \tag{11}$$

$$\mu_0\varepsilon_0 \frac{\partial^2 \boldsymbol{B}}{\partial t^2} - \nabla^2 \boldsymbol{B} = 0 \tag{12}$$

For simplicity, let us consider, say, equation (11) in one dimension only, so for the x dimension we can write:

$$\frac{\partial^2 \boldsymbol{E}}{\partial x^2} = \mu_o\varepsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} \tag{13}$$

E and ***B*** are mutually perpendicular to each other and the direction of wave propagation, and are in phase with each other.

The changing magnetic field creates a changing electric field through Faraday's law. In turn, that electric field creates a changing magnetic field through Maxwell's addition to Ampere's law (equation 6). It is this perpetual cycle that allows a self-sustaining electromagnetic wave to propagate through space.

Electromagnetism basics

Let us now look for a solution to (13) in the form of a sinusoidal wave, with speed ν and wavelength λ. Such a wave can be described by the expression:

$$\boldsymbol{E} = \boldsymbol{E}_0 \sin\left(2\pi \frac{x - \nu t}{\lambda}\right) \tag{14}$$

Differentiating (14) twice with respect to x and t separately, we get

$$\frac{\partial^2 \boldsymbol{E}}{\partial x^2} = -\boldsymbol{E}_o \left(\frac{2\pi}{\lambda}\right)^2 \sin\left(2\pi \frac{x - \nu t}{\lambda}\right) \tag{15}$$

and

$$\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = -\boldsymbol{E}_o \left(\frac{2\pi v}{\lambda}\right)^2 \sin\left(2\pi \frac{x - \nu t}{\lambda}\right) \tag{16}$$

Electromagnetism basics

Substituting (15) and (16) back into the wave equation, (14), we see that we have a solution to (13), provided that:

$$\nu^2 = \frac{1}{\mu_o \varepsilon_o} \tag{17}$$

Using the known values of the physical constants $\mu_0 = 4\pi \times 10^{-7} Hm^{-1}$ and $\varepsilon_0 = 8.854187817 \times 10^{-12} Fm^{-1}$ we can use (17) to calculate the velocity of electromagnetic wave propagation through a vacuum as:

$$\nu = 2.99792458 \cdots \times 10^8 \, m \, s^{-1} \tag{18}$$

Maxwell observed that this velocity happens to be the same as the experimentally measured speed of light, *c*, and thereby concluded that light is itself an electromagnetic wave.

Electromagnetism basics

In materials with relative permittivity ε_r and relative permeability μ_r, the speed of the electromagnetic wave becomes:

$$\nu_{\mathrm{p}} = \frac{1}{\sqrt{\mu_0 \mu_{\mathrm{r}} \varepsilon_0 \varepsilon_{\mathrm{r}}}} \tag{19}$$

Which is always less than the speed of light in a vacuum since both μ_{r} and ε_{r} are always greater than unity for real materials.

The wave described by (11) and (12) propagates through space in the positive z direction and is called a *uniform plane wave*, since it has uniform (constant) properties in a plane perpendicular to the direction of propagation. For the uniform plane wave described by (11) and (12) the plane of uniformity is the xy plane, since the direction of propagation is the z direction.

Electromagnetism basics

- The velocity referred to in (17) called the *phase velocity*. This is not the velocity of any physical entity, but the velocity at which an observer would have to move to see always a constant phase.
- It is instructive to consider the ratio of electric and magnetic field magnitudes, which has the units of Ω, i.e. :

$$Z_0 = \frac{\boldsymbol{E}}{\boldsymbol{H}} = \mu_0 c_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \frac{1}{\varepsilon_0 c_0} \tag{20}$$

With the free space values of μ_o and ε_o we can calculate the *impedance of free space* as being approximately $376.73031 \cdots \Omega$.

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Resistivity

Resistivity is a fundamental property of material which defines how much the material impedes the movement of electrons through it. Resistance depends on the geometry of the particular sample, but Resistivity does not.

Consider the resistance of a cylindrical wire made of a conducting material as shown in figure 3.

We intuitively understand that the resistance will increase with the length of the wire, l, and that thinner wires have higher resistance that thicker wires. We can therefore infer that :

$$R \propto \frac{\ell}{A} \qquad (21)$$

Where $A = \pi r^2$ is the cross-sectional area of the wire or radius r. It turns out that the constant of proportionality in (21) is the *resistivity*. We can then write (21) as :

$$R = \rho \frac{\ell}{A} \qquad (22)$$

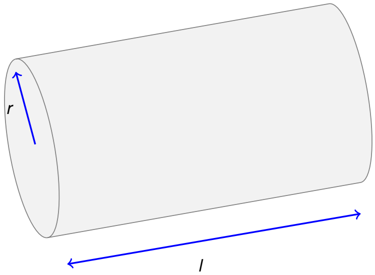

Figure 3 : Resistance of a cylindrical wire

Resistivity of common materials

Table 1 : Resistivity of typical materials (Ωm) at 300K

Material	Resistivity	Classification
Silver	1.59×10^{-8}	Conductor
Copper	1.68×10^{-8}	Conductor
Gold	2.24×10^{-8}	Conductor
Aluminium	2.65×10^{-8}	Conductor
Tungsten	5.65×10^{-8}	Conductor
Iron	9.71×10^{-8}	Conductor
Platinum	10.6×10^{-8}	Conductor
Nichrome (Ni,Fe,Cr alloy)	100×10^{-8}	Conductor
Germanium(intrinsic)	$4,700$	Semi-Conductor
Silicon(intrinsic)	2.3×10^{7}	Semi-Conductor
Gallium Arsenide (intrinsic)	10^{8}	Semi-Conductor
Glass	10^{11} to 10^{15}	Insulator
Quartz(fused)	7.5×10^{17}	Insulator
PTFE (Teflon)	10^{23} to 10^{25}	Insulator

The Skin Effect

- There are a number of peculiar physical effects which alter the behaviour of common materials and components as the frequency of operation increases.
- One of the most important is the *Skin Effect* which describes the fact that alternating current tends to accumulate near the surface of a solid conductor at higher frequencies.
- This effectively limits the cross-sectional area of the conductor with a corresponding increase in the resistance of that conductor above what would be expected at DC.

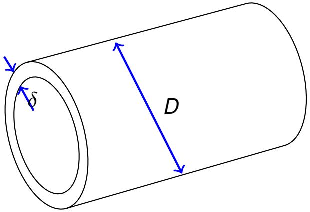

Figure 4 : The skin effect

The Skin Effect

As the frequency increases, the current density in a conductor, J, decreases exponentially from its value at the surface J_S according to the depth d from the surface according to the following relationship :

$$J = J_S\, e^{-d/\delta} \tag{23}$$

where δ is the skin depth. The skin depth is thus defined as the depth below the surface of the conductor at which the current density has fallen by a factor of $1/e$ (about 0.37) of J_S. In normal cases the skin depth can be approximated by[6]:

$$\delta = \sqrt{\frac{2\rho}{\omega \mu_r \mu_0}} \tag{24}$$

Or alternatively:

$$\delta = \frac{1}{\sqrt{\pi f \mu_r \mu_0 \sigma}} \tag{25}$$

where $\rho = 1/\sigma$ is the resistivity of the conductor and ω is the angular frequency.

The Skin Effect

A long cylindrical conductor such as the wire shown in figure 4, having a diameter D large compared to the skin depth, δ, will have a resistance approximately that of a hollow tube with wall thickness δ carrying direct current. Using a material of resistivity ρ we then find the AC resistance of a wire of length l to be :

$$R \approx \frac{\rho l}{\pi(D-\delta)\delta} \tag{26}$$

In case $\delta << D$, (26) can be approximated by :

$$R \approx \frac{L\rho}{\pi D\delta} \tag{27}$$

Table 2 : Skin depth of common materials

Material	**f=60Hz**	**f=1MHz**	**f=1GHz**
Copper	8.61mm	0.067mm	2.11 μm
Iron	0.65mm	5.03 μm	0.016 μm
Sea water	32.5m	0.25m	7.96 mm

Relative Permeability

Permeability is a physical constant that defines how much a material responds to a magnetic field. Permeability is defined as :

$$\mu = \frac{\boldsymbol{B}}{\boldsymbol{H}} \tag{28}$$

where ***B*** is the magnitude of the flux density, and ***H*** is the magnitude of the magnetic field strength. The units of permeability are Henries/meter. The permeability of a vacuum is denoted by μ_o and has the value 4×10^{-7}, (approximately $1.25663706 \times 10^{-6}$).

Most materials have permeability very close to that of a vacuum. Materials that contain iron, chrome, or nickel, however, will have a higher relative permeability (μ_r). Relative permeabilities of typical materials are shown in table 3.

Permeability is an important determinant of skin depth, since the higher the relative permeability, the less an electromagnetic wave will penetrate into the material.

Relative Permeability

Table 3 : Relative permeabilities of typical materials

Material	Type	Relative permeability (μ_r)
Silver	Diamagnetic	0.99998
Lead	Diamagnetic	0.999983
Copper	Diamagnetic	0.999991
Water	Diamagnetic	0.999991
Vacuum	Nonmagnetic	1
Air	Paramagnetic	1.0000004
Aluminum	Paramagnetic	1.00002
Ferrite(nickel zinc)	Ferromagnetic	16 to 640
Cobalt	Ferromagnetic	250
Nickel	Ferromagnetic	600
Ferrite (manganese zinc)	Ferromagnetic	640
Mild Steel (0.2% C)	Ferromagnetic	2,000
Iron (0.2% impurity)	Ferromagnetic	5,000
Silicon Iron	Ferromagnetic	7,000

Relative Permittivity

- Permittivity refers to the ability of a material to polarize in response to an externally applied electric field and thereby reduce the total electric field inside the material. In other words, permittivity is a measure of a material's ability to transmit (or "permit") an electric field within it.
- As with permeability, the permittivity generally depends on the frequency of the applied field. This frequency dependence reflects the fact that a material's polarization does not respond instantaneously to an applied field.
- There is a certain delay between the application of the applied field and the response, which can be represented by a phase difference at a given frequency.

Relative Permittivity

Table 4 shows the relative permittivity (ε_r) of some common materials. The table also lists something called the *loss tangent* that will be explained in the following slides.

Table 4 : Relative permittivity (dielectric constant) of typical materials

Material	Dielectric constant (ε_r)	Loss-tangent* ($\tan \delta_\varepsilon$)
Vacuum	1.0	0
Teflon	2.1	0.0003
Nylon	2.4	0.0083
Sandy soil (dry)	2.55	0.0062
Silicon Dioxide	3.9	0.001
Thermoset polyester	4.0	0.0050
Paper	3-4	0.0125 - 0.0333
Concrete (dry)	4.5	0.0111
Glass	4 to 7	0.0050
Soda lime glass	6.0	0.02
Alumina	9.0	0.0006
RT/duroid 5870 (microstrip substrate)	2.33	0.0009

Losses in dielectric and magnetic materials

When a time-varying electric field is applied to a material, the polarization dipoles inside the material will flip back and forth in response to the field. The finite mass of the charge carriers has two important consequences:

1. Work has to be done to move them, which means that some of the applied energy will be 'lost' in the material.
2. It takes a finite time for these dipoles to move, which means that the polarization vector will lag behind the applied electric field[3].

The result of this phase lag is that permittivity should be more correctly defined as a complex number. i.e.:

$$\varepsilon = \varepsilon' - j\varepsilon'' \tag{29}$$

$$= \varepsilon' - j\frac{\sigma}{\omega} \tag{30}$$

Both the real and imaginary parts of (29) are frequency dependent[4], but at lower frequencies the imaginary part of ε' is small and is usually ignored.

The real part of (29), ε', is a measure of how much energy from an external electric field is stored in the material. The imaginary part of permittivity, ε'', is called the *loss factor* and is a measure of how dissipative or lossy a material is. The loss factor includes the effects of both dielectric loss and conductivity, σ.

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Wire

- When current is flowing through a wire, a magnetic field is induced around the wire.
- If the magnetic field is forced to expand and contract by changes in the current, a voltage will be induced in the wire that will tend to oppose the change in current flow (Faraday's law).
- This effect manifests itself as a *self-inductance*.
- A good estimate of the self-inductance of a cylindrical wire can be obtained from the following empirical formula[1]:

$$L = 0.002l \left[2.3 \ln \left(\frac{4l}{d} \right) - 0.75 \right] \mu H \tag{31}$$

where l = length of wire in cm and d = diameter of the wire in cm.

According to this formula, 5cm of wire of 1 mm diameter will have an inductance of 50 nH, which translates to the rather appreciable reactance of 314Ω at 1 GHz.

Resistors

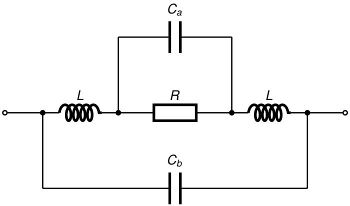

Figure 5 : High frequency equivalent circuit of a typical resistor

$$Z_{R_{equiv}} = \frac{1}{j\omega C_b + \frac{1}{j2\omega L + \frac{1}{G + j\omega C_a}}} \quad (32)$$

where the conductance $G = 1/R$.

typical values:

$$R = 1k\Omega$$
$$L = 60nH$$
$$C = 4pF$$

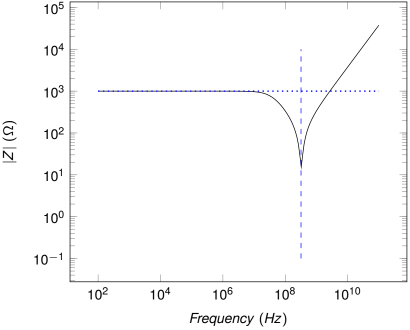

Figure 6 : $|Z|$ vs frequency for a typical resistor (nominal value: $1k\Omega$)

Capacitors

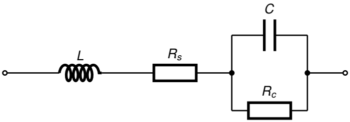

Figure 7 : High frequency equivalent circuit of a typical capacitor

The effective impedance of the circuit in figure 7 is given by:

$$Z_{C_{equiv}} = R_s + j\omega L + \frac{1}{G_c + j\omega C} \qquad (33)$$

Where $G_c = 1/R_c$.

Typical values:

$$R_s = 0.1\Omega$$
$$R_c = 100k\Omega$$
$$L = 20nH$$

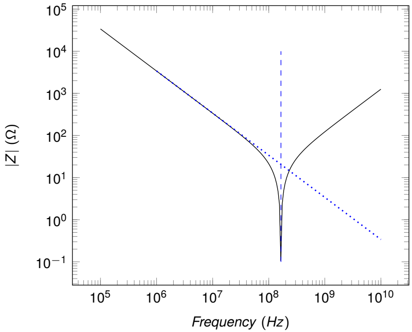

Figure 8 : $|Z|$ vs frequency for a typical capacitor (nominal value: $47pF$)

Inductors

Typical values:

$$R_s = 0.05\Omega$$
$$L = 25nH$$
$$C_s = 0.5pF$$

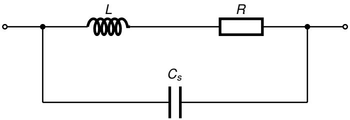

Figure 9 : High frequency equivalent circuit of a typical inductor

By inspection the impedance of the equivalent circuit in figure 9 is:

$$Z_{L_{equiv}} = \frac{R_s + j\omega L}{1 + j\omega C_s(R_s + j\omega L)} \tag{34}$$

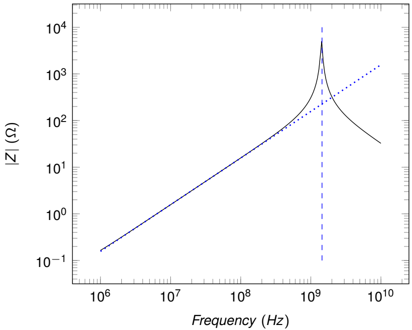

Figure 10 : $|Z|$ vs frequency for a typical inductor (nominal value: $25nH$)

Surface Mount Devices (SMD)

Table 5 : SMD two-terminal component sizes

Identifier		Dimensions		
Imperial	Metric	Imperial	Metric	Power rating (resistors)
0402	1005*	0.04 × 0.02 in	1.0 × 0.5 mm	0.1W
0504	1210*	0.05 × 0.04 in	1.2 × 1.0 mm	0.1W
0603	1508	0.06 × 0.03 in	1.5 × 0.8 mm	0.1W
0805	2012	0.08 × 0.05 in	2.0 × 1.2 mm	0.125W
1005*	2512	0.10 × 0.05 in	2.5 × 1.2 mm	0.125W
1206	3216	0.12 × 0.06 in	3.2 × 1.6 mm	0.25W
1210*	3225	0.12 × 0.10 in	3.2 × 2.5 mm	0.5W
1812	4532	0.18 × 0.12 in	4.5 × 3.2 mm	0.75W
2512	6332	0.25 × 0.13 in	6.4 × 3.2 mm	1.0W

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Maximum power transfer

- A lot of RF and microwave design involves moving signal power most efficiently from one place to another.
- The maximum transfer of signal power implies that losses in the signal path should be minimised.
- This requires that the source and load impedances be *matched*.
- As a starting point, it is worth briefly reviewing the DC maximum power transfer theorem which will be familiar to the reader from basic circuit theory.

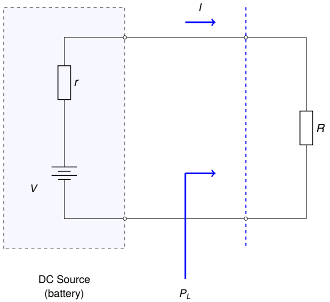

Figure 11 : Maximum power transfer : DC case

Maximum power transfer

$$P_L = I^2 R = \left(\frac{V}{R+r}\right)^2 R \qquad (35)$$

If we consider how load power, P_L, changes with load resistance, R, we can write :

$$\frac{dP_L}{dR} = V^2 \left[\frac{(R+r)^2 - 2R(R+r)}{(R+r)^4}\right] \qquad (36)$$

Load power is maximised when the above derivative equals 0, i.e. when :

$$(R+r)^2 = 2R(R+r) \qquad (37)$$

Solving (37) yields the familiar condition for maximum power transfer in the DC case, namely :

$$R = r \qquad (38)$$

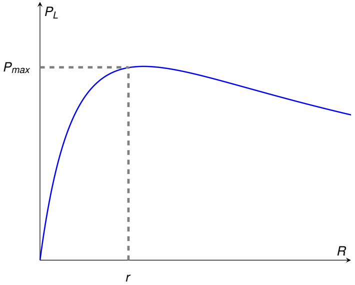

Figure 12 : Load power versus load resistance : DC case

Maximum power transfer

When operating at radio frequencies we need to take into account the reactive elements of both load and source. We therefore have the situation shown in figure **??** for complex source and load impedances, Z_S and Z_L respectively.

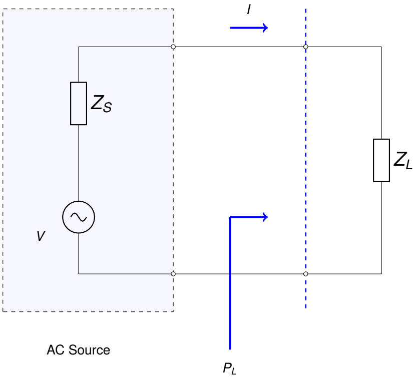

Figure 13 : Maximum power transfer : AC case

Maximum power transfer

We proceed by matching the resistive and reactive elements of Z_S and Z_L independently. Matching of the resistive elements requires that (38) be satisfied, as in the DC case. In addition, the reactive elements of source and load need to cancel each other out (i.e. be of equal magnitude but opposite sign). We therefore have the following requirement for maximum power transfer in the RF case :

$$R_S + jX_S = R_L - jX_L \quad (39)$$

Or, more simply :

$$Z_S = Z_L^* \quad (40)$$

Where $*$ indicates the complex conjugate.

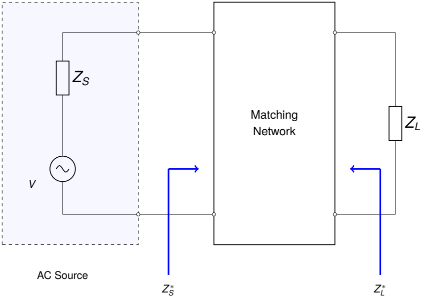

Figure 14 : Maximum power transfer : with impedance matching

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Reflection coefficient

Given a "normalised" arbitrary impedance, $z = r + jx$, we can define the reflection coefficient as[7]:

$$\Gamma = \frac{z-1}{z+1} = \frac{(r-1)+jx}{(r+1)+jx} \quad (41)$$

and conversely :

$$z = \frac{1+\Gamma}{1-\Gamma} \quad (42)$$

In terms of normalised admittance, i.e. $y = g + jb$ where $y = 1/z$ we can write:

$$\Gamma = \frac{1-y}{1+y} = \frac{(1-g)-jb}{(1+g)+jb} \quad (43)$$

and conversely:

$$y = \frac{1-\Gamma}{1+\Gamma} \quad (44)$$

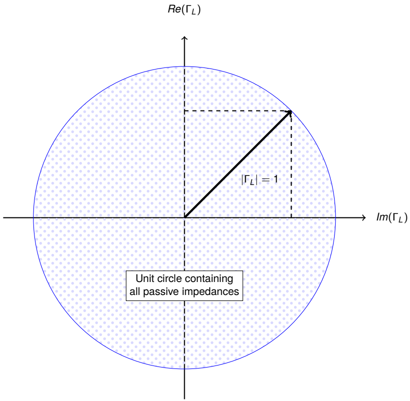

Figure 15 : Cartesian coordinate system on the reflection coefficient (Γ) plane

Return loss

Return loss is the loss of power in the signal returned/reflected by a *discontinuity*. A discontinuity typically occurs where there is a mismatch between two impedances in a transmission line system.
Return loss is usually expressed in decibels (dB) as a log of the ratio of two powers, as follows:

$$RL_{dB} = 10 \log_{10} \frac{P_i}{P_r} \tag{45}$$

The relationship between reflection coefficient and return loss can be found by applying the definition of reflection coefficient:

$$RL_{dB} = -10 \log |\Gamma|^2 = -20 \log |\Gamma| \tag{46}$$

Description	**Load Impedance**	$\lvert\Gamma\rvert$	**% of power reflected**	**VSWR**	**Return Loss**
Perfect match	Z_o	0	0	1.00	$-\infty$
Fairly good match	$\simeq Z_o$	0.03	0.1%	1.06	$-31dB$
Poor match	$2 \times Z_o$ or $0.5 \times Z_o$	0.33	11%	2.00	$-9.5dB$
Very poor match	$10 \times Z_o$ or $0.1 \times Z_o$	0.82	67%	10.00	$-1.7dB$
Open circuit	∞	1.00	100%	∞	0
Short circuit	0	1.00	100%	∞	0

dBm

- Due to the extremely wide range of power levels in microwave systems, ranging from picowatts (10^{-12} W) to gigawatts (10^{9} W), it is more convenient to represent power on a logarithmic scale in units of 'dBm', being defined as the ratio of the power level, in milliwatts, to one milliwatt, i.e. :

$$P(dBm) = 10\log_{10}\left(\frac{P(mW)}{1mW}\right) \quad (47)$$

- One milliwatt is chosen as the reference level simply because many of the systems of interest are operating with powers of this order of magnitude.
- Any powers quoted in dBm are assumed to be referenced relative to a 50 Ω system impedance. If this is not the case then it needs to be stated.

Table 6 : dBm versus milliwatts

Power (mW)	Power (dBm)
1 μW	-30 dBm
5 μW	-23dBm
10 μW	-20 dBm
0.1 mW	-10 dBm
0.5 mW	-3 dBm
1 mW	0 dBm
2 mW	3 dBm
10 mW	10 dBm
100 mW	20 dBm
200 mW	23 dBm
1 W	30 dBm
1 kW	60 dBm

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Quality factor, Q

Energy loss occurs in all real passive components, including reactive components such as capacitors and inductors.

These components can be represented as resonant circuits at microwave frequencies, where energy is being exchanged back and forth between inductive and capacitive energy storage elements of the equivalent circuit and some of the energy is being 'lost' in parasitic resistances.

Given the presence of real-world losses, we can define a measure of *Quality*, or "Q" for any such equivalent circuit as follows :

$$Q = \frac{\textit{energy stored}}{\textit{average power dissipated}} \tag{48}$$

Higher Q indicates a lower rate of power loss in the circuit relative to the energy stored.

The stored energy is the sum of energies stored in all the lossless reactive elements (inductors and capacitors), whereas the energy dissipated is the sum of the energies lost in all the resistive elements per cycle. In other words, a circuit containing only ideal reactive elements would have an infinite Q.

Real world components and circuits all exhibit some electrical losses and therefore have a finite Q. In the case of individual reactive components such as capacitors and inductors, the higher the Q the closer the component approaches the ideal.

Q of series combinations

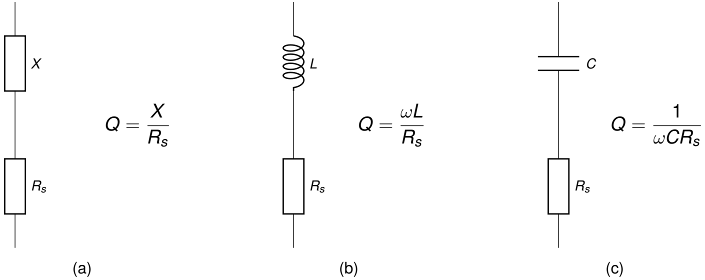


Figure 16 : Q of series combinations

The peak energy stored in the inductor in figure 16(b) is $\frac{1}{2}LI^2$, whereas the energy dissipated in the resistor in one cycle, T, is equal to $\frac{1}{2}I^2R_sT = \frac{1}{2}I^2R_s(1/f_o)$. Hence:

$$Q_s = 2\pi \frac{\frac{1}{2}LI_{max}^2}{\frac{1}{2}I_{max}^2 R\,(1/f)} = \frac{\omega_o L}{R} \quad (49)$$

For figure 16(c) an equivalent expression to (49) as :

$$Q_s = 2\pi \frac{\frac{1}{2}I_{max}^2/\omega_o^2 C}{\frac{1}{2}I_{max}^2 R_s\,(1/f)} = \frac{1}{\omega_o C R_s} \quad (50)$$

Q of parallel combinations

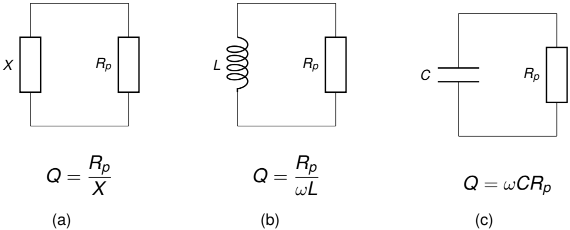
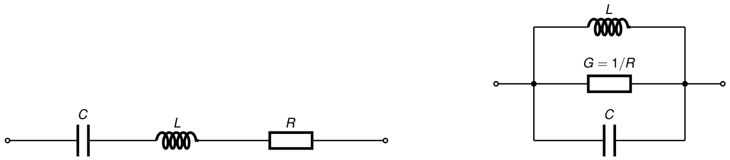


Figure 17 : Q of parallel combinations

Q of RLC circuits

We will now extend the above analysis to the series LCR circuit shown in figure 18(a).

- At resonance the stored electrical energy in the series *RLC* circuit is being exchanged back and forth between the inductor and the capacitor once each cycle.
- When the energy stored in the capacitor is at a maximum, the energy stored in the inductor is zero and vice versa.



(a) Series RLC circuit (b) Parallel RLC circuit

Figure 18 : RLC circuits

Q of RLC circuits

We can therefore use either (49) or (50) to calculate the Q of a series RLC circuit, as both these expressions will yield the same result, so for the circuit in figure 18(a) we have :

$$Q_s = \frac{\omega_o L}{R} = \frac{1}{\omega_o C R} \qquad (51)$$

At resonance, the net reactance of a series RLC circuit is zero, so the impedance of the circuit in figure 18(a) at resonance is simply R, which is the lowest value of impedance obtainable for this circuit at any frequency.

The parallel RLC circuit of figure 18(b) can be considered as the dual of figure 18(a). In other words the net susceptance is zero at resonance so the admittance is simply $1/R$, which is the lowest value of admittance obtainable for this circuit at any frequency.

Q and resonance

- A graph of Y/Y_o versus frequency is shown in figure 19.
- The *half-power* points are indicated on figure 19, being the frequencies, labelled ω_1 and ω_2, either side of ω_o, where Y/Y_o falls to a value of 0.707 of its value at resonance.
- The distance between ω_1 and ω_2 is called the *half-power* bandwidth of the circuit, or $\Delta\omega$.
- These parameters are related to Q by the following, which also serves as another working definition of Q for a resonant circuit[5]

:

$$Q = \frac{\omega_o}{\Delta\omega} = \frac{f_o}{\Delta f} \tag{52}$$

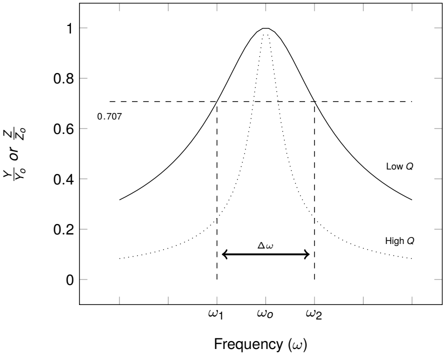


Figure 19 : Resonant frequency and Q

Q of series versus parallel circuits

For every series combination in figure 16, at a given frequency, there is an equivalent parallel combination, and vice versa. The ability to convert between series and parallel representations is important in network design. This is accomplished as follows:
Consider the circuit of figure 16(a). We can write the admittance as :

$$
\begin{aligned}
Y &= \frac{1}{R_s + jX} && (53)\\
&= \frac{R_s - jX}{R_s^2 + X^2} && (54)\\
&= \frac{R_s}{R_s^2 + X^2} - j\frac{X}{R_s^2 + X^2} && (55)
\end{aligned}
$$

Which can be rewritten as :

$$
Y = \frac{1/R_s}{R_s\left[1 + \left(\frac{X}{R_s}\right)^2\right]} + \frac{1/X}{j\left[\left(\frac{R_s}{X}\right)^2 + 1\right]} \quad (56)
$$

In terms of the Q of the series circuit of figure 16(a), Q_s, we can write :

$$
Y = \frac{1}{R_s(1 + Q_s^2)} + \frac{1}{jX\left[\frac{1}{Q_s^2} + 1\right]} \quad (57)
$$

Q of series versus parallel circuits

So, from (57) we can see that the series combination of figure 16(a) is equivalent to the parallel combination in figure 17(a), where the parallel elements (conductance and susceptance) are :

$$G = \frac{1}{R_s(1+Q_s^2)} \quad (58)$$

$$B = \frac{-Q}{R_s(1+Q_s^2)} \quad (59)$$

Since the conductance, G, in (58) is the reciprocal of the equivalent parallel resistance, R_p, we can state the following relationship between R_s and R_p :

$$R_p = R_s(1+Q_s^2) \quad (60)$$

Which implies that R_p will always be greater than R_s for any $Q_s > 0$, as one would expect. Re-arranging (60), we obtain the following useful formula for the Q of a resonant circuit in terms of series and parallel resistances that will pop up repeatedly in different forms throughout this book and especially in chapter **??**, where we introduce matching network design:

$$Q_s = \sqrt{\left(\frac{R_p}{R_s}\right) - 1} \quad (61)$$

Loaded Q and External Q

- So far we have analysed RLC circuits in isolation, but in the real world these circuits are usually embedded in a larger system.
- At the very least, our *RLC* circuit must be connected to a source and a load, both of which will contain their own resistive elements.
- When a resonant circuit is connected to the outside world, the total losses, for the purpose of calculating Q, will have to include losses in the source and load resistances.
- A typical situation where a parallel *RLC* resonant circuit is connected to a source, R_S and load, R_L, is shown in figure 20.

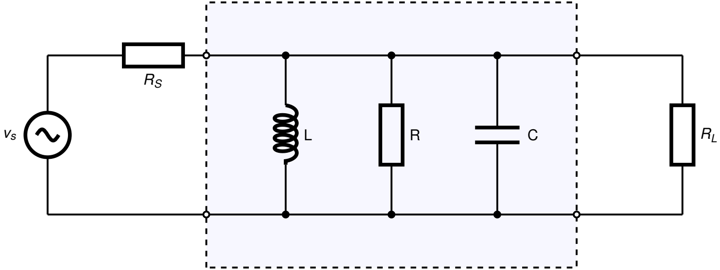


Figure 20 : Parallel RLC circuit with source and load

Loaded Q and External Q

The loaded Q may be determined by considering the overall Q of the entire circuit with external lossy elements taken into account, in other words, considering only the energy stored in the capacitor :

$$Q_L = \frac{\textit{total susceptance}}{\textit{total conductance}} = \frac{\omega_o C}{G_{total}} = \frac{\omega_o C}{G_s + G + G_L} \tag{62}$$

Where : $G_s = 1/R_s$, $G = 1/R$ and $G_L = 1/R_L$,

- We could use an equivalent expression by considering only the energy stored in the inductor and the results will be the same. We can immediately see, from (62), that any finite values of G_S and G_L will have the effect of reducing the overall value of Q.
- This concurs with our intuitive understanding that the loaded Q of a passively loaded circuit can never be higher than its unloaded Q, since the addition of external passive circuit elements can only serve to add losses.
- Another way of considering loaded Q is to introduce the concept of *external* Q, which represents the effect of all the elements that are not intrinsic to the 'unloaded' resonant circuit.
- Unloaded Q_u and external Q_e are combined together to give the loaded Q_L of the circuit according to the following general equation[2]:

$$\frac{1}{Q_L} = \frac{1}{Q_u} + \frac{1}{Q_e} \tag{63}$$

Q of a one-port resonator

The input impedance of the resonator in figure 21 at a single frequency, ω, is given by:

$$Z_{in} = R + j\omega L - j\frac{1}{\omega C} \qquad (64)$$

Which can be re-written in terms of the resonant frequency, ω_o, as :

$$Z_{in} = R + j\hat{Z}_o\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right) \qquad (65)$$

where $\hat{Z}_o$ is a parameter we will refer to as the *characteristic impedance* of the resonator defined by:

$$\hat{Z}_o = \sqrt{\frac{L}{C}} \qquad (66)$$

and ω_o is the nominal resonant frequency defined by:

$$\omega_o = \frac{1}{\sqrt{LC}} \qquad (67)$$

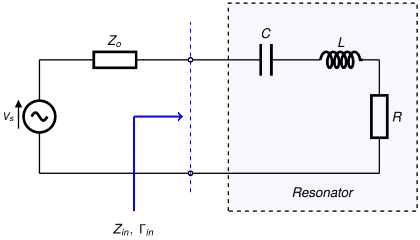


Figure 21 : One-port RLC resonator

Q of a one-port resonator

We have written Z_{in} in the form of (65) in order to emphasise the fact that Z_{in} reduces to simply R when $\omega = \omega_o$. The input reflection coefficient of the resonator of figure 21 at the frequency, ω, can be calculated from (41) as :

$$\Gamma_{in} = \frac{Z_{in} - Z_o}{Z_{in} + Z_o} \tag{68}$$

Applying (65) we get:

$$\Gamma_{in} = \frac{R - Z_o + j\hat{Z}_o\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)}{R + Z_o + j\hat{Z}_o\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)} \tag{69}$$

$$\Gamma_{in} = \frac{\left(\frac{R - Z_o}{\hat{Z}_o}\right) + j\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)}{\left(\frac{R + Z_o}{\hat{Z}_o}\right) + j\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)} \tag{70}$$

Q of a one-port resonator

The various Q factors for the circuit in figure 21 are defined as :

Unloaded Q :

$$Q_u = \frac{\omega_o L}{R} = \frac{1}{\sqrt{LC}} \cdot \frac{L}{R} = \sqrt{\frac{L}{C}} \cdot \frac{1}{R} = \frac{\hat{Z}_o}{R} \tag{71}$$

External Q :

$$Q_e = \frac{\omega_o L}{Z_o} = \frac{1}{\sqrt{LC}} \cdot \frac{L}{Z_o} = \sqrt{\frac{L}{C}} \cdot \frac{1}{Z_o} = \frac{\hat{Z}_o}{Z_o} \tag{72}$$

Loaded Q :

$$Q_L = \frac{\omega_o L}{(R + Z_o)} = \frac{Q_u}{(1 + Z_o/R)} = \frac{RQ_u}{(R + Z_o)} = \frac{\hat{Z}_o}{(R + Z_o)} \tag{73}$$

Q of a one-port resonator

By combining (71), (72), (73) with (63) we can now write (70) in terms of the various Q factors as follows:

$$\Gamma_{in} = \frac{\left(\frac{1}{Q_u} - \frac{1}{Q_e}\right) + j\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)}{\left(\frac{1}{Q_u} + \frac{1}{Q_e}\right) + j\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)} \tag{74}$$

- Equation (74) is a universal expression for the input reflection coefficient of any microwave resonator in terms of the resonant frequency and he two Q factors Q_u and Q_e.
- The utility of equation (74) lies in the fact that these parameters can usually be more easily measured than the fundamental components R, L and C, especially in the case of metal cavities and dielectric resonators, where R, L and C do not exist as discrete physical entities.

From (74), the squared magnitude of the input reflection coefficient is given by :

$$|\Gamma_{in}(\omega)|^2 = \frac{\left(\frac{1}{Q_u} - \frac{1}{Q_e}\right)^2 + \left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)^2}{\left(\frac{1}{Q_u} + \frac{1}{Q_e}\right)^2 + \left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)^2} \tag{75}$$

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